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Things Are Not Always as They Appear

Solving Rational Equations

LEARNING GOAL

In this lesson, you will:

- Solve rational equations in one variable.

KEY TERMS

- rational equation
- extraneous solution

A paradox is a statement that leads to a contradiction. Consider the following statements:

"Don't go near the water until you have learned how to swim!"

"Nobody shops at that store anymore, it's always too crowded."

They involve faulty logic. You couldn't actually learn to swim if you never got near water, and obviously a lot of people still shop at that store. Some paradoxes use mathematics that lead you to a solution path that does not necessarily make sense in real life. A famous example is Zeno's paradox that involves traveling a distance approaching a specific value, but never quite getting there. Here is an example of Zeno's paradox:

Suppose you are walking to catch a stationary bus that is 20 meters away. You decide to get there by going half the distance every few seconds. This means that you walk 10 meters, then 5 meters, 2.5 meters, 1.25 meters, and so on until you reach the bus.

The paradox is that if you continually halve the distance between you and the bus, you will never actually reach the bus.

Have you ever followed the correct steps to solve a problem, but then your answer didn't make sense?

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PROBLEM 1 Method Mayhem

A **rational equation** is an equation that contains one or more rational expressions. You have already solved simple rational equations with a single variable as a denominator and performed simple operations using rational expressions. You will follow the same rules and guidelines when solving more involved rational equations.

There are multiple methods you can use to solve rational equations. Depending on the structure of the equation some methods will be more efficient than others.

1. Randall and Sully solved the equation $\frac{x+5}{x+2} = \frac{x+1}{x-5}$.

Randall

$$\frac{x+5}{x+2} = \frac{x+1}{x-5}$$

Restrictions: $x \neq -2, 5$

$$\cancel{(x+2)}(x-5)\cdot\frac{x+5}{\cancel{x+2}} = (x+2)\cancel{(x-5)}\cdot\frac{x+1}{\cancel{x-5}}$$

$$(x-5)(x+5) = (x+2)(x+1)$$

$$x^2 - 25 = x^2 + 3x + 2$$

$$-25 = 3x + 2$$

$$-27 = 3x$$

$$x = -9$$

Sully

$$\frac{x+5}{x+2} = \frac{x+1}{x-5}$$

Restrictions: $x \neq -2, 5$

$$(x+5)(x-5) = (x+2)(x+1)$$

$$x^2 - 25 = x^2 + 3x + 2$$

$$-25 = 3x + 2$$

$$-27 = 3x$$

$$x = -9$$

a. Explain Randall's method of solving.

b. Sully used proportional reasoning to solve the equation. Explain how he solved the equation.

c. Which method do you prefer for this problem? Explain your reasoning.

2. Sully was presented with a slightly different equation to solve. Notice that a new factor appears in one of the denominators. Again, he uses proportional reasoning to solve.

Sully

$$\frac{x+5}{(x-5)(x+2)} = \frac{x+1}{x-5}$$

Restrictions: $x \neq 5, -2$

$$(x-5)(x+2)(x+1) = (x+5)(x-5)$$
$$(x^2+2x-5x-10)(x+1) = x^2-25$$
$$x^3-3x^2-10x+x^2-3x-10 = x^2-25$$
$$x^3-3x^2-13x+15 = 0$$

5	1	−3	−13	15
	↓	5	10	−15
	1	2	−3	0

−3	1	−3	−13	15
	↓	−3	18	−15
	1	−6	5	0

1	1	−3	−13	15
	↓	1	−2	−15
	1	−2	−15	0

$p = \pm 15, \pm 5, \pm 3, \pm 1$

$q = \pm 1$

Possible rational roots $\left(\frac{p}{q}\right)$: $\pm 15, \pm 5, \pm 3, \pm 1$

Using synthetic division, I realize the three roots are 5, −3, and 1. However, from my list of restrictions, I know that $x \neq 5$. So, my solutions to the equation can only be $x = -3$ or $x = 1$. I will check to see if they work.

Check $x = -3$

$$\frac{-3+5}{(-3-5)(-3+2)} = \frac{-3+1}{-3-5}$$
$$\frac{2}{(-8)(-1)} = \frac{-2}{-8}$$
$$\frac{2}{8} = \frac{2}{8}$$

Check $x = 1$

$$\frac{1+5}{(1-5)(1+2)} = \frac{1+1}{1-5}$$
$$\frac{6}{(-4)(3)} = \frac{2}{-4}$$
$$-\frac{6}{12} = -\frac{2}{4}$$
$$-\frac{1}{2} = -\frac{1}{2}$$

Thus, $x = -3$ and $x = 1$.

a. What is different about the structure of this equation compared to the equation in Question 1?

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b. Prior to checking his solution, explain why Sully identified three possible roots to the equation?

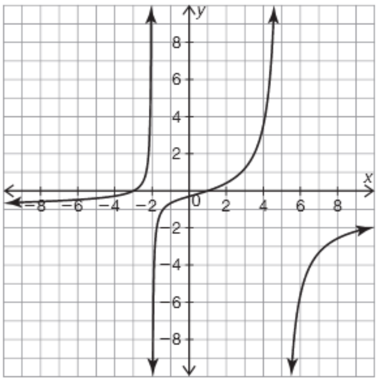
c. Use Randall's method to solve the second equation Sully solved.

d. Which method is more efficient based on the structure of the original equation? Explain your reasoning.

There is a mathematical reason why Sully determined an extra solution to his equation. Recall that one of the basic principles of algebra is that you can multiply both sides of an equation by a non-zero real number or expression, as long as you maintain balance to the equation. When you multiply both sides of the equation by an expression that contains a variable, you may introduce solutions that were not there before. Notice Sully multiplied by $(x - 5)$. By doing so, he introduced an additional solution. These extra solutions are called *extraneous solutions*. **Extraneous solutions** are solutions that result from the process of solving an equation; but are not valid solutions to the equation.

3. Sully wanted to graph the equation using a graphing calculator. He rewrote the equation so that one side of the equation equaled zero. Then, he graphed:

$y_1 = \frac{x + 1}{x - 5} - \frac{x + 5}{(x - 5)(x + 2)}$. Consider the graph:

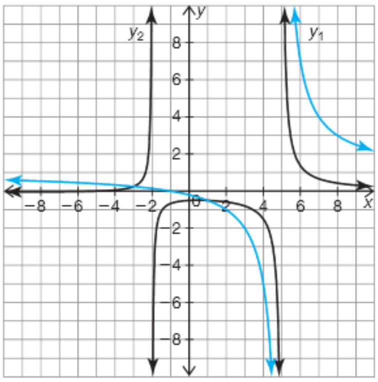


Use the graph to explain why $x \neq 5$, $x = 3$ and $x = -1$.

4. Mike used a graphing calculator to solve the same equation:

$y_1 = \frac{x + 1}{x - 5}$

$y_2 = \frac{x + 5}{(x - 5)(x + 2)}$.



How does Mike's graph represent the solutions to the equation?

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5. Analyze the methods Jake and Sasha used to solve $\frac{2x+4}{x^2-2x-8} = \frac{x+1}{x-4}$.

Sasha

$$\frac{2x+4}{(x-4)(x+2)} = \frac{x+1}{x-4} \quad x \neq 4, -2$$

$$\frac{2x+4}{(x-4)(x+2)} = \frac{x+1}{x-4} \cdot \frac{(x+2)}{(x+2)}$$

$$\frac{2x+4}{(x-4)(x+2)} = \frac{x^2+3x+2}{(x-4)(x+2)}$$

$$\cancel{(x-4)(x+2)}\left[\frac{2x+4}{\cancel{(x-4)(x+2)}} = \frac{x^2+3x+2}{\cancel{(x-4)(x+2)}}\right]$$

$2x + 4 = x^2 + 3x + 2$

$x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0$

$x = -2, 1$

I know that $x \neq -2$, so I just need to check $x = 1$.

Check $x = 1$

$$\frac{2(1)+4}{(1-4)(1+2)} \stackrel{?}{=} \frac{1+1}{1-4}$$

$$\frac{6}{(-3)(3)} \stackrel{?}{=} \frac{2}{-3}$$

$$-\frac{2}{3} = -\frac{2}{3}$$

Jake

$$\frac{2x+4}{x^2-2x-8} = \frac{x+1}{x-4}$$

$$\frac{2(x+2)}{(x-4)(x+2)} = \frac{x+1}{x-4}$$

$x \neq 4, -2$

$$\frac{2\cancel{(x+2)}}{(x-4)\cancel{(x+2)}} = \frac{x+1}{x-4}$$

$$\cancel{(x-4)}\left[\frac{2}{\cancel{x-4}} = \frac{x+1}{\cancel{x-4}}\right]$$

$2 = x + 1$

$x = 1$

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a. Describe how Jake and Sasha's methods are similar and different.

b. What would happen if you tried to solve this problem using proportional reasoning

6. Analyze the methods Seth, Damiere, and Sidonie each used to solve the equation $\frac{6}{x^2 - 4x} + \frac{4}{x} = \frac{2}{x - 4}$.

Seth

$$\frac{6}{x^2 - 4x} + \frac{4}{x} = \frac{2}{x - 4}$$

$$\frac{6}{x(x - 4)} + \frac{4}{x} \cdot \frac{(x - 4)}{(x - 4)} = \frac{2}{x - 4} \cdot \frac{x}{x}$$

$x \neq 0, 4$

$$\frac{6}{x(x - 4)} + \frac{4x + 16}{x(x - 4)} = \frac{2x}{x(x - 4)}$$

$$\frac{6 + 4x - 16}{x(x - 4)} = \frac{2x}{x(x - 4)}$$

$$\cancel{x(x - 4)}\left[\frac{6 + 4x - 16}{\cancel{x(x - 4)}} = \frac{2x}{\cancel{x(x - 4)}}\right]$$

$$6 + 4x - 16 = 2x$$

$$4x - 10 = 2x$$

$$2x = 10$$

$$x = 5$$

Sidonie

$$\frac{6}{x^2 - 4x} + \frac{4}{x} = \frac{2}{x - 4}$$

$$\frac{6}{x(x - 4)} + \frac{4}{x} = \frac{2}{x - 4}$$

$x \neq 0, 4$

$$(x(x - 4)) \cdot \left[\frac{6}{x(x - 4)} + \frac{4}{x} = \frac{2}{x - 4}\right]$$

$$6 + 4(x - 4) = 2x$$

$$6 + 4x - 16 = 2x$$

$$4x - 10 = 2x$$

$$2x = 10$$

$$x = 5$$

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Damiere

$$\frac{6}{x(x-4)} + \frac{4}{x} = \frac{2}{x-4}$$

$x \neq 0, 4$

$$\frac{6}{x^2-4x} + \frac{4}{x} = \frac{2}{x-4}$$

$$\frac{(x)(x-4)}{(x)(x-4)} \cdot \frac{6}{x^2-4x} + \frac{(x^2-4x)(x-4)}{(x^2-4x)(x-4)} \cdot \frac{4}{x} = \frac{2}{x-4} \cdot \frac{(x^2-4x)(x)}{(x^2-4x)(x)}$$

$$\frac{6x^2-24x}{x^4-8x^3+16x^2} + \frac{4x^3-16x^2-16x^2+64x}{x^4-8x^3+16x^2} = \frac{2x^3-8x^2}{x^4-8x^3+16x^2}$$

$$\cancel{x^4-8x^3+16x^2} \cdot \frac{6x^2-24x+4x^3-16x^2-16x^2+64x}{\cancel{x^4-8x^3+16x^2}} = \cancel{x^4-8x^3+16x^2} \cdot \frac{2x^3-8x^2}{\cancel{x^4-8x^3+16x^2}}$$

$6x^2 - 24x + 4x^3 - 16x^2 - 16x^2 + 64x = 2x^3 - 8x^2$

$4x^3 - 26x^2 + 40x = 2x^3 - 8x^2$

$2x^3 - 18x^2 + 40x = 0$

$2x(x^2 - 9x + 20) = 0$

$2x(x-4)(x-5) = 0$

$x = 0, 4, 5$

I know that $x \neq 0, 4$, so I will just check $x = 5$.

Check:

$x = 5$

$$\frac{6}{5^2-4(5)} + \frac{4}{5} = \frac{2}{5-4}$$

$$\frac{6}{25-20} + \frac{4}{5} = \frac{2}{1}$$

$$\frac{6}{5} + \frac{4}{5} = 2$$

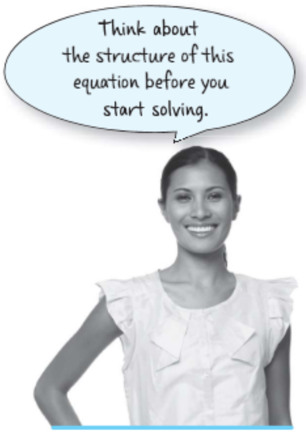
$$\frac{10}{5} = 2$$

$$2 = 2$$

a. Describe the methods Seth, Sidonie, and Damiere each used to solve the rational equation.

b. Prior to checking his solution, explain why Damiere identified three possible roots to the equation.

7. Solve the equation $\frac{10}{x^2 - 2x} + \frac{1}{x} = \frac{3}{x - 2}$. Explain why you chose your solution method.



PROBLEM 2 Seeing Structure

1. Cut out each of the equations on the following three pages. Before solving each equation, think about how the structure of the equation informs your solution method. Then, sort the equations based on the solution method you intend to use. Finally, solve each equation. Be sure to list the domain restrictions.

a. $\frac{12}{x+5} = -2$

b. $\frac{x-5}{3} = \frac{x-38}{12} - \frac{x}{4}$

c. $\frac{x^2-5x}{4} = \frac{8x}{2}$

d. $\frac{1}{x-5} = \frac{5}{x^2+2x-35}$

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e. $\frac{3}{x-1} + \frac{2}{5x+5} = \frac{-3}{x^2+1}$

f. $\frac{x-5}{x-2} = \frac{8}{9}$

g. $\frac{5}{x} = 25 + \frac{5}{x}$

h. $\frac{1}{x^2} + \frac{1}{x} = \frac{1}{2x^2}$

i. $\frac{-2}{x+3} + \frac{3}{x-2} = \frac{5}{x^2+x-6}$

j. $\frac{7}{x+3} = \frac{8}{x-2}$

k. $\frac{x+3}{x^2-1} + \frac{-2x}{x-1} = 1$

l. $\frac{3}{x^2+2x} = \frac{6}{x^2}$

Paste your solved rational equations in the space provided.

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2. Did you solve the equations using the method you first intended?

Be prepared to share your solutions and methods.